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We explore the quantum wave nature from the Newtonian mechanics by usuing a concept: velocity field. At first, we retwirte the relativistic Newton's second law as a field equation in terms of the velocity field. Next, we show that the Dirac equation is derived from the field equation in a rigorous and consistent manner.

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In the past century, many attempts were made to explore the origin of quantum wave nature, the topic attracts attention continuously, for the fundamentals of quantum mechanics remain to be unclear so far in a sense. In this paper, we propose a concept: velocity field, then show that the Dirac equation may be deriving from the relativistic Newton's second law in terms of the velocity field in a rigorous manner, the work gives a reasonable origin for the quantum mechanics.

According to the Newtonian mechanics, in a hydrogen atom, the single electron revolves in an orbit about the nucleus, its motion can be described with its position in an inertial Cartesian coordinate System $S : (x_1, x_2, x_3, x_4 = ict)$. As the time elapses, the electron draws a spiral path (or orbit) in imagination, as shown in Fig.1(a).

If the reference frame S rotates through an angle about the x_2-axis, becomes a new reference frame S', there is the Lorentz transformation linking the frames S and S'. Then in the frame S', the spiral path of the electron tilts with respect to the x_4'-axis with the angle as shown in Fig.1(b). At one moment, for example, $t_4' = t_0$ moment, the spiral path pierces many points at the plane $t_4' = t_0$, for example, the points labeled a, b and c in Fig.1(b), these points indicate that the electron can appear at many points at the time t_0, in agreement with the concept of the probability in quantum mechanics. This situation gives us a hint for deriving quantum wave nature from the Newtonian mechanics.

Because the electron pierces the plane $t_4' = t_0$ with the 4-velocity u, for every pierced point we can label a local 4-velocity vector. The pierced points may be numerous if the path winds up itself into a cell about the nucleus (due to a nonlinear effect in a sense), then the 4-velocity vectors for the pierced points form a 4-velocity field. It is noted that the observation plane selected for the piercing can be taken at an arbitrary orientation, so the 4-velocity field may be expressed in general as $u(x_1, x_2, x_3, x_4 = ict)$, i.e. the velocity u is of a function of position.

At every point in the reference frame S' the electron satisfies the relativistic Newton's second law

$$m\frac{du_\mu}{d\tau} = qF_{\mu\nu}u_\nu \tag{1}$$

the notations consist with the convention [1]. Since the reference frame is a Cartesian coordinate system whose axes are orthogonal to one another, there is no distinction between covariant and contravariant components, only subscripts need be used. Here and below, summation over twice repeated indices is implied in all case, Greek indices will take on the values 1,2,3,4, and regarding the mass m as a constant. As the mention above, the 4-velocity u can be regarded as a 4-velocity vector field, then

$$\frac{du_\mu}{d\tau} = \frac{\partial u_\mu}{\partial x_\nu}\frac{\partial x_\nu}{\partial \tau} = u_\nu \partial_\nu u_\mu \tag{2}$$

$$qF_{\mu\nu}u_\nu = qu_\nu(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{3}$$

Substituting them back into Eq.(1), and re-arranging their terms, we obtain

$$\begin{aligned} u_\nu\partial_\nu(mu_\mu + qA_\mu) &= u_\nu\partial_\mu(qA_\nu) \\ &= u_\nu\partial_\mu(mu_\nu + qA_\nu) - u_\nu\partial_\mu(mu_\nu) \\ &= u_\nu\partial_\mu(mu_\nu + qA_\nu) - \frac{1}{2}\partial_\mu(mu_\nu u_\nu) \\ &= u_\nu\partial_\mu(mu_\nu + qA_\nu) - \frac{1}{2}\partial_\mu(-mc^2) \\ &= u_\nu\partial_\mu(mu_\nu + qA_\nu) \end{aligned} \tag{4}$$

Using the notation

$$K_{\mu\nu} = \partial_\mu(mu_\nu + qA_\nu) - \partial_\nu(mu_\mu + qA_\mu) \tag{5}$$

Eq.(4) is given by

$$u_\nu K_{\mu\nu} = 0 \tag{6}$$

Because $K_{\mu\nu}$ contains the variables $\partial_\mu u_\nu$, $\partial_\mu A_\nu$, $\partial_\nu u_\mu$ and $\partial_\nu A_\mu$ which are independent from u_ν, then a solution satisfying Eq.(6) is of

$$K_{\mu\nu} = 0 \tag{7}$$

$$\partial_\mu(mu_\nu + qA_\nu) = \partial_\nu(mu_\mu + qA_\mu) \tag{8}$$

The above equation allows us introduce a potential function Φ in mathematics, further set $\Phi = -i\hbar \ln \psi$, we obtain a very important equation

$$(mu_\mu + qA_\mu)\psi = -i\hbar\partial_\mu\psi \tag{9}$$

We think it as an extended form of the relativistic Newton's second law in terms of 4-velocity field. ψ representing the wave nature may be a complex mathematical function, its physical meanings will be determined from experiments after the introduction of the Planck's constant $\hbar$.

Multiplying the two sides of the following familiar equation by ψ

$$-m^2c^4 = m^2u_\mu u_\mu \tag{10}$$

which stands at every points in the 4-velocity field, and using Eq.(9), we obtain

$$\begin{aligned}
-m^2c^4\psi &= mu_\mu(-i\hbar\partial_\mu - qA_\mu)\psi \\
&= (-i\hbar\partial_\mu - qA_\mu)(mu_\mu\psi) - [-i\hbar\psi\partial_\mu(mu_\mu)] \\
&= (-i\hbar\partial_\mu - qA_\mu)^2\psi - [-i\hbar\psi\partial_\mu(mu_\mu)]
\end{aligned} \tag{11}$$

According to the continuity condition for the electron motion

$$\partial_\mu(mu_\mu) = 0 \tag{12}$$

we have

$$-m^2c^4\psi = (-i\hbar\partial_\mu - qA_\mu)^2\psi \tag{13}$$

It is known as the Klein-Gordon equation.

On the condition of non-relativity, the Schrodinger equation can be derived from the Klein-Gordon equation [2](P.469).

However, we must admit we are careless when we use the continuity condition Eq.(12), because, from Eq.(9) we obtain

$$\partial_\mu(mu_\mu) = \partial_\mu(-i\hbar\partial_\mu \ln\psi - qA_\mu) = -i\hbar\partial_\mu\partial_\mu \ln\psi \tag{14}$$

where we have used the Lorentz gauge condition. Thus from Eq.(10) to Eq.(11) we obtain

$$-m^2c^4\psi = (-i\hbar\partial_\mu - qA_\mu)^2\psi + \hbar^2\psi\partial_\mu\partial_\mu \ln\psi \tag{15}$$

This is of a perfect wave equation for describing accurately the motion of the electron. In other wards, The Klein-Gordon equation is ill for using the mistaken continuity condition Eq.(12). Comparing with the Dirac equation result, we find that the last term of Eq.(15) corresponds to the spin effect of electron. In the following we shall show the Dirac equation from Eq.(9) and Eq.(10).

In general, there are many wave functions which satisfy Eq.(9) for the electron, these functions and corresponding momentum components are noted with $\psi(j)$and $P_\mu(j) = mu_\mu(j)$, respectively, where $j = 1, 2, 3, ..., N$, then Eq.(10) can be given by

$$
\begin{aligned}
0 &= P_\mu(j)P_\mu(j)\psi^2(j) + m^2c^4\psi^2(j) \\
&= \delta_{\mu\nu}P_\mu(j)\psi(j)P_\nu(j)\psi(j) + mc^2\psi(j)mc^2\psi(j) \\
&= (\delta_{\mu\nu} + \delta_{\nu\mu})P_\mu(j)\psi(j)P_\nu(j)\psi(j)(\mu \geq \nu) \\
&\quad + mc^2\psi(j)mc^2\psi(j) \\
&= 2\delta_{\mu\nu}P_\mu(j)\psi(j)P_\nu(j)\psi(j)(\mu \geq \nu) \\
&\quad + mc^2\psi(j)mc^2\psi(j) \\
&= 2\delta_{\mu\nu}\delta_{jk}\delta_{jl}P_\mu(k)\psi(k)P_\nu(l)\psi(l)(\mu \geq \nu) \\
&\quad + \delta_{jk}\delta_{jl}mc^2\psi(k)mc^2\psi(l)
\end{aligned} \tag{16}
$$

where δ is the Kronecker delta function, $j, k, l = 1, 2, 3, ..., N$. Here, specially, we do not take j sum over; P represents momentum, not operator. Suppose there are two matrices a and b which satisfy

$$a_{\mu jk}a_{\nu jl} + a_{\nu jk}a_{\mu jl} = 2\delta_{\mu\nu}\delta_{jk}\delta_{jl} \tag{17}$$

$$a_{\mu jk}b_{jl} + b_{jk}a_{\mu jl} = 0 \tag{18}$$

$$b_{jk}b_{jl} = \delta_{jk}\delta_{jl} \tag{19}$$

then Eq.(16) can be rewritten as

$$
\begin{aligned}
0 &= (a_{\mu jk}a_{\nu jl} + a_{\nu jk}a_{\mu jl})P_\mu(k)\psi(k)P_\nu(l)\psi(l)(\mu \geq \nu) \\
&\quad + (a_{\mu jk}b_{jl} + b_{jk}a_{\mu jl})P_\mu(k)\psi(k)mc^2\psi(l) \\
&\quad + b_{jk}b_{jl}mc^2\psi(k)mc^2\psi(l) \\
&= [a_{\mu jk}P_\mu(k)\psi(k) + b_{jk}mc^2\psi(k)] \\
&\quad \cdot [a_{\nu jl}P_\nu(l)\psi(l) + b_{jl}mc^2\psi(l)] \\
&= [a_{\mu jk}P_\mu(k)\psi(k) + b_{jk}mc^2\psi(k)]^2
\end{aligned} \tag{20}
$$

Consequently, we obtain a wave equation:

$$a_{\mu jk}P_\mu(k)\psi(k) + b_{jk}mc^2\psi(k) = 0 \tag{21}$$

There are many solutions for a and b which satisfy Eq.(17-19), we select a familiar set of a and b as [2]:

$$N = 4 \tag{22}$$

$$a_n = [a_{n\mu\nu}] = \begin{bmatrix} 0 & \sigma_n \\ \sigma_n & 0 \end{bmatrix} = \alpha_n \tag{23}$$

$$a_4 = [a_{4\mu\nu}] = I \tag{24}$$

$$b = [b_{jk}] = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \beta \tag{25}$$

where α_n are the Pauli spin matrices, $n = 1, 2, 3$. Substituting them into Eq.(21), we obtain

$$[(-i\hbar\partial_4 - qA_4) + \alpha_n(-i\hbar\partial_n - qA_n) + \beta mc^2]\psi = 0 \tag{26}$$

where ψ is an one-column matrix about $\psi(k)$. Eq.(26) is known as the Dirac equation.

Of course, on the condition of non-relativity, the Schrodinger equation can be derived from the Dirac equation [2](P.479).

It is noted that Eq.(26), Eq.(21), Eq.(16) and Eq.(15) are equivalent despite they have the different forms, because they all originate from Eq.(9) and Eq.(10).

We do not know exactly what kind of path of the electron in a hydrogen atom is, so the illustration of Fig.1 is an imaginary one for visualizing the motion of the electron. But we know that the electron path will pierce many points at any observation time plane like $t'_4 = t_0$ for any arbitrary reference frame S' if the path or orbit exists in the 4-dimensional space-time, the points may be numerous. Therefore there is a 4-velocity field for the motion of the electron. The 4-velocity field is a key concept for our deduction.

It follows from Eq.(9) that the path of a particle is analogous to "lines of electric force" in 4-dimensional space-time.. In the case that the Klein-Gordon equation stands, i.e. Eq.(12) stands, at any point, the path can have but one direction (i.e. the local 4-velocity direction), hence only one path can pass through each point of the space-time.. In other words, the path never intersects itself when it winds up itself into a cell about the nucleus. No path originates or terminates in the space-time.. But, in general, the divergence of the 4-velocity field does not equal to zero, as indicated in Eq.(14), so the Dirac equation would be better than the Klein-Gordon equation in accuracy.

The present work focus on the formalism and pursuing the correction and strictness in mathematics, its interpretation in physical terms remains to be discussed further in the future.

In conclusion, the path of the electron of a hydrogen atom should wind up itself into a cell about the nucleus in 4-dimensional space-time, therefore there is a 4-velocity field for describing the motion of the electron. In terms of the 4-velocity field, the relativistic Newton's second law can be rewritten as a wave field equation. By this discovery, the Klein-Gordon equation, Schrodinger equation and Dirac equation can be derived from the Newtonian mechanics on different conditions, respectively.

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FIG. 1. The motion of the electron of a hydrogen atom in 4-dimensional space-time in imagination.

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Fig.1 The motion of the electron of a hydrogen atom in 4-dimensional space-time in imagination.